countable or locally compact space must be a k-space in this sense (compare [4], p. 231, Theorem 13); a space in which each compact subset is locally compact is a k-space if and only if it is a quotient of a locally compact space (compare [1], Theorem 11.9.4); a quotient of any k-space is a k-space.

10. Specialize when necessary. In some situations T_2 may be needed or convenient. In such cases one specializes; but this is no reason to use T_2 as a blanket assumption from the beginning, any more than prospective use of the Čech compactification justifies dealing only with completely regular spaces from the beginning.

11. Examples. Some examples of useful non-Hausdorff topologies are: the topology induced by a non-separating family of seminorms on a vector space, the one point compactification (see [9]), the Zariski topology used by algebraic geometers.

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MATHEMATICAL NOTES

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ON DIVIDING A SQUARE INTO TRIANGLES

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Sometime ago in this MONTHLY, Fred Richman and John Thomas [1] asked the following puzzling question:

Can a square S be divided into an odd number of nonoverlapping triangles T_i, all of the same area?

In [2], the answer was shown to be no, provided $S=[0, 1]\times[0, 1]$ and the coordinates of the vertices of the T_i are rational numbers with odd denominators. In this note we shall show that the answer is always no. In fact we shall prove the following more general result.

Suppose that $S=[0,1]\times[0,1]$ *is divided into* m *nonoverlapping triangles* T_i; *let* $a_i=$ area T_i. *Then there is a polynomial* f *with integer co-efficients such that* $f(a_1, \cdots, a_m)=1/2$.

There are two parts to the proof: one combinatorial, the other valuation theoretic. The combinatorial argument generalizes an argument made in [2]. By itself it may be made to prove the desired result when the vertices of the T_i all have rational coordinates. But to handle the case of arbitrary vertices, it becomes necessary to argue with "congruences mod 2 in the reals." This is where valuation theory comes in; we make use of absolute values on the reals extending the 2-adic absolute value of the rationals. In the course of the proof the theorem of the extension of valuations plays a remarkable and unexpected role.

We begin with the combinatorial argument. Let R be a region in the plane bounded by a simple closed polygon. Suppose R is divided into m nonoverlapping triangles T_i. By a *vertex* we shall mean a vertex of some T_i, by a *face* a face of some T_i or of R. Two vertices are called *adjacent* if they are in the same face and the line segment joining them contains no other vertices. A *basic segment* is a line segment joining two adjacent vertices. Note that the boundary of each T_i is a union of nonoverlapping basic segments; the same is true of the boundary of R. Suppose now that the vertices are divided into three disjoint sets, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. We shall say that a face or a basic segment is of *type* $\mathcal{AB}$ if it has one endpoint in $\mathcal{A}$ and one in $\mathcal{B}$.

LEMMA. *Suppose that no face contains vertices of all three types and that* R *has an odd number of faces of type* $\mathcal{AB}$. *Then some* T_i *has vertices of all three types.*

To prove the lemma note the following. A face of type $\mathcal{AB}$ contains an odd number of basic segments of type $\mathcal{AB}$, while a face not of type $\mathcal{AB}$ contains an even number of basic segments of type $\mathcal{AB}$. (Use the fact that a face contains vertices of at most two types to prove this.) Suppose that no T_i has vertices of all three types. Then each T_i has either 0 or 2 faces of type $\mathcal{AB}$. Hence the boundary of T_i contains an even number of basic segments of type $\mathcal{AB}$. Similarly, the boundary of R contains an odd number of basic segments of type $\mathcal{AB}$. But this is impossible; in an obvious sense the boundary of R is congruent to the sum of the boundaries of the T_i modulo 2.

We now come to the valuation theoretic part of the proof and need to introduce some further terminology. Let K be a field. By an *ultranorm* (sometimes called a *non-Archimidean absolute value*) on K we mean a function $\|\ \|$ from K to the nonnegative real numbers satisfying:

$$\|xy\| = \|x\|\cdot\|y\| \tag{1}$$

$$\|x+y\| \leqq \max(\|x\|, \|y\|) \tag{2}$$

$$\|x\| = 0 \Leftrightarrow x = 0. \tag{3}$$

We can easily prove that $\|1\|=\|-1\|=1$, and that equality holds in equation (2) unless $\|x\|=\|y\|$.

As an example let K be the field of rational numbers. Any $x \neq 0$ in K may be written as $2^t(r/s)$, where r and s are odd integers and t is an integer. Set $\|x\| = (1/2)^t$. In this way we get an ultranorm on the rationals in which $\|2\| < 1$. This ultranorm is known as the *2-adic absolute value*. The more general fact that we shall need is this: *There is an ultranorm on the field of real numbers (or more generally on any extension of the rational numbers) such that* $\|2\| < 1$. This follows from the theorem of the extension of valuations whose proof may be found in many places; for example see [**3**].

Granting the above facts we may argue as follows. Choose an ultranorm on the reals for which $\|2\| < 1$. Divide the points of the plane into three sets in the following way:

(1) (x, y) is in $\mathcal{A}$ if $\|x\| < 1$ and $\|y\| < 1$,

(2) (x, y) is in $\mathcal{B}$ if $\|x\| \geqq 1$ and $\|x\| \geqq \|y\|$,

(3) (x, y) is in $\mathcal{C}$ if $\|y\| \geqq 1$ and $\|y\| > \|x\|$.

Suppose now that $P = (x, y)$ and $P' = (x', y')$ are points and that P' is a translate of P by a point of type $\mathcal{A}$; in other words, that both $\|x' - x\| < 1$ and $\|y' - y\| < 1$. *Then P and P' have the same type.* If P is of type $\mathcal{A}$ this is obvious. If P is of type $\mathcal{B}$, then $\|x'\| = \|x\| \geqq 1$, while $\|y'\| \leqq \max(1, \|y\|) \leqq \|x\| = \|x'\|$; so P' is of type $\mathcal{B}$ too. If P is of type $\mathcal{C}$ the argument is similar.

It is now easy to see that a line L cannot contain points of all three types. For by translating a point of type $\mathcal{A}$ on L to the origin we may assume that $(0, 0)$ is on L. Let (x, y) and (x', y') be points of L of types $\mathcal{B}$ and $\mathcal{C}$. Then $\|x\| \geqq \|y\|$, $\|y'\| > \|x'\|$, and $\|xy'\| > \|x'y\|$. This is absurd as $xy' = x'y$. Note also that if a triangle $\boldsymbol{T}$ has vertices of all three types then $\|\text{area } \boldsymbol{T}\| > 1$. For we may assume that the vertex of $\boldsymbol{T}$ of type $\mathcal{A}$ is $(0, 0)$. Let (x, y) and (x', y') be the vertices of types $\mathcal{B}$ and $\mathcal{C}$. Then area $\boldsymbol{T}$, up to sign, is equal to $\frac{1}{2}(xy' - x'y)$. But $\|xy'\| > \|x'y\|$. So $\|\text{area } \boldsymbol{T}\| = \|\frac{1}{2}\| \, \|xy'\| = \|\frac{1}{2}\| \cdot \|x\| \cdot \|y'\| > 1$.

Suppose now that $\boldsymbol{S} = [0, 1] \times [0, 1]$ is divided into m nonoverlapping triangles $\boldsymbol{T}_i$ each of area $1/m$. Obviously $\boldsymbol{S}$ has exactly one face of type $\mathcal{A}\mathcal{B}$; by the lemma, some $\boldsymbol{T}_i$ has vertices of all three types. By the paragraph above, $\|\text{area } \boldsymbol{T}_i\| = \|1/m\| > 1$. So m is even. (Note that if all vertices have rational coordinates, then we can argue directly with the 2-adic absolute value of the rationals, and avoid the theorem of the extension of valuations; this is essentially what was done in [2].)

Finally, we indicate the proof of the more general theorem mentioned at the beginning of this paper. Let $\boldsymbol{A}$ be the ring $\boldsymbol{Z}[a_1, \cdots, a_m]$. If 2 generates the unit ideal in $\boldsymbol{A}$, then $1 = 2f(a_1, \cdots, a_m)$ and we are done. If $2\boldsymbol{A} \neq \boldsymbol{A}$, then 2 is contained in a height 1 prime ideal $\boldsymbol{P}$ of $\boldsymbol{A}$. The integral closure of the local ring of $\boldsymbol{P}$ on $\boldsymbol{A}$ is a discrete valuation ring. This ring gives rise to an ultranorm on the quotient field of $\boldsymbol{A}$ such that $\|a_i\| \leqq 1$, while $\|2\| < 1$. Extend this ultranorm to the reals, and use it to subdivide the plane into points of three types as above. Then, as above, some $\boldsymbol{T}_i$ has vertices of all three types, and $\|a_i\| = \|\text{area } \boldsymbol{T}_i\| > 1$, a contradiction. (By using valuation rings instead of ultranorms we could simplify the proof a little.)

The above proof is not so wildly nonconstructive as it first appears. For the entire argument is carried out in the field generated by the coordinates of the vertices. So it is only necessary to extend our ultranorm from Q to this finitely generated field, not to the entire field of real numbers.

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SYLVESTER'S PROBLEM ON COLLINEAR POINTS AND A RELATIVE

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In 1893, J. J. Sylvester posed the following question in the Educational Times: *Given a finite set of points in the plane, not all lying on a straight line, must there be a line containing exactly two of the points?*

This question was not settled until after 1930, when T. Gallai proved the affirmative answer. Many proofs have been given since that time, some sparked by a restatement of the problem by P. Erdös [3] in this MONTHLY. V. C. Williams [9] recently gave a proof in this MONTHLY. Extensive references to the literature which has grown up around the problem can be found in [2], [4], and [5].

If we settle upon the projective plane as an appropriate setting and state the problem in its dual form, we obtain: *Given n lines in the projective plane, not all concurrent, must there be a point lying on exactly two of the lines?*

Thinking of the projective plane as a sphere with antipodes identified, and then immediately observing that the identification contributes nothing to the problem, we obtain the following equivalent formulation of Sylvester's problem: *Given n great circles on a sphere, not all concurrent, must there be a point lying on exactly two of the great circles?*

In his exquisite paper on zonohedra, H. S. M. Coxeter [1] observes in passing that in this context the positive answer to Sylvester's question follows immediately from the fact that there exists no map on the sphere (each of whose countries has at least 3 sides) such that all vertices have valence 6 or more. For a set of n great circles such that no point lies on exactly two circles, would form a map on the sphere such that each vertex has valence at least 6.

Here is a short proof of this fact about maps. Suppose we are given a map on the sphere such that each face (country) has 3 or more sides. Suppose each vertex had valence at least 6 (the valence of a vertex is the number of edges emanating from that vertex). If V denotes the total number of vertices, E the number of edges, and F the number of faces, then we have Euler's relation,

$$V - E + F = 2. \tag{1}$$